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A NEW APPROACH FOR ANALYTIC SOLUTION OF IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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Abstract: Impulsive delay differential equations are extensively used to study the dynamical behavior and mathematical modeling of many problems in engineering, population growth, disease epidemics, drugs administration, automation, traffic flow, earthquake protection etc. But many impulsive delay differential equations cannot be solved analytically or it is very difficult to solve due to the discontinuity at impulse moments. In this paper, an algorithm for the analytic solution of impulsive delay differential equations with impulses at fixed moments is presented. The diagrammatic representation of this method is illustrated with the help of an example. Further the gotten solution is compared with numerical solution and it is also exponentially stable.

Keywords and Phrases: Impulsive Delay Differential equations, Analytic solutions.

2010 Mathematics Subject Classification: 34D20, 34K20, 34C60.

1. Introduction

Many fields such as remote control dynamical systems, drugs administration, population dynamics, automation, traffic flow, earthquake protection etc. used the system of delay differential equations for their representation. Many of them face a change of state abruptly due to short term disturbances whose duration is negligible in comparison with the duration of the whole process. Such systems are generally represented by Impulsive delay differential equations. Although Impulsive delay differential equations are more natural and useful in such cases. A large number of practical problems concerning with impulsive delay differential still need to be explored for solutions. Many researchers worked on the numerical and analytic solutions of partial and ordinary differential equations and their stability [3,6-9,12-14]. Some solutions regarded to impulsive differential equations are done analytically. Some of the famous researchers like V. Lakshmikantham, D. Bainov, and many others presented significance results in last few decades [1-14]. However many impulsive delay differential equations cannot be solved analytically or it is very difficult to solve them because of the dependence on some previous values.

The algorithm proposed in this paper is interpreted according to the theory of impulsive differential equations written by V. Lakshmikantham et. al [1]. This method has widen the scope of application of many Impulsive delay differential equations which are not solved analytically or solved approximately by using numerical techniques. This method converts impulsive delay differential equation on a given interval into ordinary differential equation on that particular interval by using the known history function of that interval. Such ordinary differential equations are solved and if any impulse is encountered then we can consider the effect of impulses at these moments as given in (2). The diagrammatic representation of the solution obtained with the help of numerical simulation as discussed in [3,12,14] shows that the solution is exponentially stable, which means that it is also amenable for further treatment. At last, this analytic solution is also compared with the corresponding numerical solution of the same problem obtained by using Euler method as suggested by N.Hamzah [14] and can be observed that both the solutions are moving in same direction and are approximately same.

This paper is organized as follows. In section 1, we presented some notations and definitions. In sections 2, numerical algorithm to find the analytic solution of impulsive delay differential equations is given. Numerical solution of the problem by using Euler method with diagram is given in section 3. At last concluding remarks are given.

1.1 Preliminaries

Let R^n denotes the n-dimensional real space. For given constant $\tau > 0$, we define the following class of functions:
$PC([-\tau, 0], R^n) = \{\psi : [-\tau, 0] \to R^n; \psi(t)$ is continuous everywhere except at finite number of points at which it is right continuous and left limit exists$\}$.

Consider the linear impulsive delay differential system:

$$\begin{aligned} x'(t) &= f(t, x_t) = a(t)x(t) + b(t)x(t-\tau), && t \geq t_0,\ t \neq t_k \\ \triangle x(t_k) &= I_k(x(t_k^-)) \\ x_{t_0}(r) &= \psi(r), && r \in [-\tau, 0] \end{aligned} \tag{1}$$

where $f : R_+ \times PC([-\tau, 0], R^n) \rightarrow R^n$; $I_k \in C[R^n, R^n]$; $\psi \in PC([-\tau, 0], R^n)$; $0 \leq t_0 < t_1 < t_2 < ... < t_k < ...$, with $t_k \rightarrow \infty$ as $k \rightarrow \infty$; $\triangle x(t) = x(t^+) - x(t^-)$; $a(t)$ and $b(t)$ be the functions of t and $x_t \in PC([-\tau, 0], R^n)$ are defined by $x_t(r) = x(t + r)$ for $r \in [-\tau, 0]$.

1.2 Solution of impulsive differential equations

Any set of functions $\phi_i(t),\ i = 1, 2, 3, ...n$ is said to be a solution of impulsive differential equation (1), if it satisfies the system of differential equations along with conditions of jump and initial conditions. A problem of existence and uniqueness of solutions of impulsive differential equation (1) is reduced to that corresponding ordinary differential equations i.e. $x'(t) = f(t, x)$. Let $x(t)$ be the solution of impulsive differential equation (1) which satisfies the initial condition $x_(t_0) = x_0$. Also let Ω^+, Ω^- be the maximal intervals on which the solution can be continued to the right and left of t_0 respectively. Then the next expression is valid:

$$x(t) = \begin{cases} x_0 + \displaystyle\int_{t_0}^{t} f(s, x_t(s))ds + \sum_{t_0 < t_k < t} I_k(x(t_k)), & \text{for } \ t \in \Omega^+, \\ \\ x_0 + \displaystyle\int_{t_0}^{t} f(s, x_t(s))ds - \sum_{t_0 < t_k < t} I_k(x(t_k)), & \text{for } \ t \in \Omega^-, \end{cases} \tag{2}$$

2. Algorithm for analytic solution

This method converts impulsive delay differential equation on a given interval into ordinary differential equation on that particular interval by using the known history function of that interval. Such ordinary differential equations are solved and if any impulse is encountered then we can consider the effect of impulses at these moments as given in (2). This process can be repeated for the next interval with newly found solution as the history function for the next interval. Consider the impulsive delay differential equation (1) along with initial condition at time $t = t_0$ be given. Let I_k be impulsive operators that act at the impulse moments $t = t_k, k \in Z$ and $x(t)$ be the solution whose value is to be determined. Now, algorithm for the solution is as:

1. On the interval $[-\tau, 0]$, the function x_t is the given function ψ, which is known by initial condition. So the equation can be solved in this interval and say $x^0(t)$ be the solution

2. In interval $[0, \tau]$, the given system reduces to
   $x'(t) = a(t)x(t) + b(t)x^0(t-\tau)$
   $x(0) = \psi(0)$
   Now $t \in [0, \tau]$ implies $t - \tau \in [-\tau, 0]$, so $x(t-\tau)$ becomes $x^0(t-\tau)$ on interval $[0, \tau]$. So above equation is not a delay equation as $x^0(t-\tau)$ is a known. Thus equation can be solved by using initial condition at $t = 0$.

3. At the jump point $t = t_k$, we apply the impulsive operators to find the values of the right limit. This can be repeated until the next impulse moment is encountered. Denote the solution of this interval $[0, \tau]$ as $x^1(t)$.

4. Similarly, in interval $[\tau, 2\tau]$ the given system becomes
   $x'(t) = a(t)x(t) + b(t)x^1(t-\tau)$
   $x(\tau) = \psi(\tau)$
   Which is again an ordinary differential equation free from any delay. So we can solve this with the initial condition at $t = \tau$ for the interval $[\tau, 2\tau]$.

5. This process can be continued for subsequent intervals.

2.1 Analytical Solution of Numerical Example

Consider the differential system:

$$\begin{aligned} x'(t) &= -(2+t^2)x(t) + t^2 x(t-1), \qquad\qquad t \geq t_0, t \neq t_k \\ x(t_k) &= \left(1 + \frac{1}{2^k}\right)(x(t_k^-)) \end{aligned}$$

along with the initial condition

$$x_{t_0} = \psi \begin{cases} 0, & t \in [-1, 0] \\ 1.7 & t = 0 \end{cases}$$

We applied here the above mentioned method to solve this impulsive delay differential equation with impulses at fixed moments as discussed in the above mentioned algorithm. The analytic solution so obtained is as follows:

$$x(t) = \begin{cases} 0 \quad \text{for } t \in [-1, 0], \\ (1.7)e^{-2t-\frac{t^3}{3}} \qquad \text{for } t \in [0, 1], \\ (1.7)\left(\left(\frac{t^5}{5} - \frac{t^4}{4} + \frac{2t^3}{3}\right) + 2.200555556\right) e^{-2t-\frac{t^3}{3}} \quad \text{for } t \in [1, 2], \\ (1.7)\begin{pmatrix} \left(\frac{t^{10}}{50} - \frac{29t^9}{180} + \frac{319t^8}{480} - \frac{5t^7}{3} + \frac{101t^6}{36} - \frac{957t^5}{300} + \frac{537t^4}{240} - \frac{57t^3}{90}\right) \\ +2.200555556\left(\frac{t^5}{5} - \frac{t^4}{4} + \frac{2t^3}{3}\right) - 45.954935 \end{pmatrix} e^{-2t-\frac{t^3}{3}} \\ \hfill \text{for } t \in [2, 3], \end{cases}$$

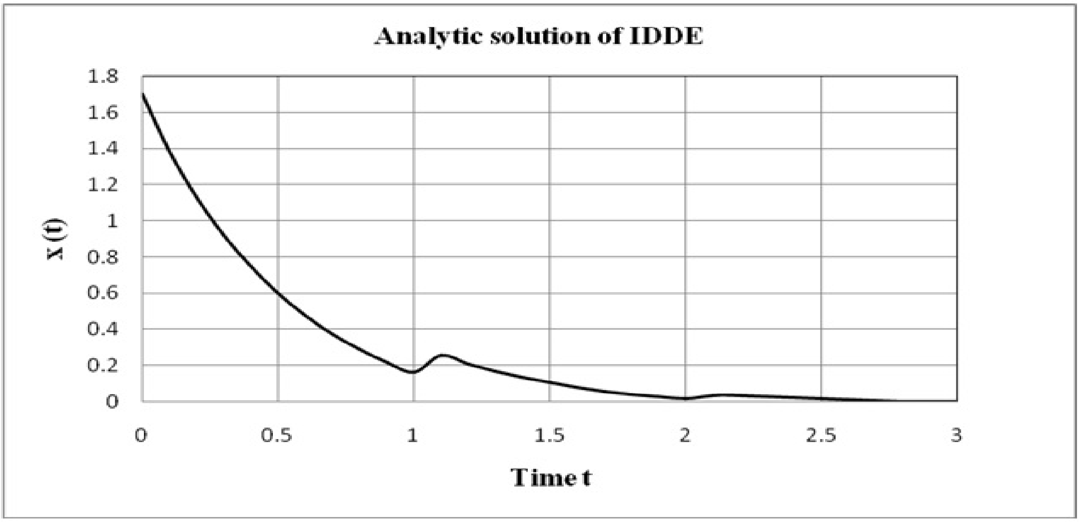

Figure 1. Diagramatic representation of the analytic solution of IDDE

Remark 1. It can be observed from the Figure 1 that analytic solution so obtained have sudden changes at impulsive moments, which shows the effect of impulsive forces. Moreover, the gotten solution is exponentially stable which increases the reliability and usability of the solution for further treatment.

3. Numerical Solution by using Eulers method

Again consider the same impulsive delay differential system as:

$$x'(t) = -(2 + t^2)x(t) + t^2 x(t-1), \qquad t \geq t_0, t \neq t_k$$

$$x(t_k) = \left(1 + \frac{1}{2^k}\right)(x(t_k^-))$$

along with the initial condition $x_{t_0} = \psi \begin{cases} 0, & t \in [-1, 0] \\ 1.7 & t = 0 \end{cases}$

Now, we will apply the usual Euler method for Impulsive differential system to get the numerical solution as follows:

1. Initially, take $t_0 = 0$ and set $x(t_0) = 1.7$ and $x(t_0 - 1) = 0$ as given in the initial condition.

2. Apply Eulers method to get the next values of $x(t)$ up to the first impulse moment as follows:
$x(t_{i+1}) = x(t_1) + h\{-(2 + t_i^2)x(t_i) + t_i^2 x(t_i - 1)\}, \quad t \geq t_0, \ t \neq t_k.$

3. At impulsive moment i.e. when $t = t_k$, then impulsive operator acts and brings rapid changes in solution as
$x(t_k) = \left(1 + \frac{1}{2^k}\right)(x(t_k^-)).$

4. Repeat the step 2 and step 3 till the next impulse moment and then apply impulsive operator again and again at every impulsive moment.

5. The above process is repeated until the desired values of $x(t)$ is obtained.

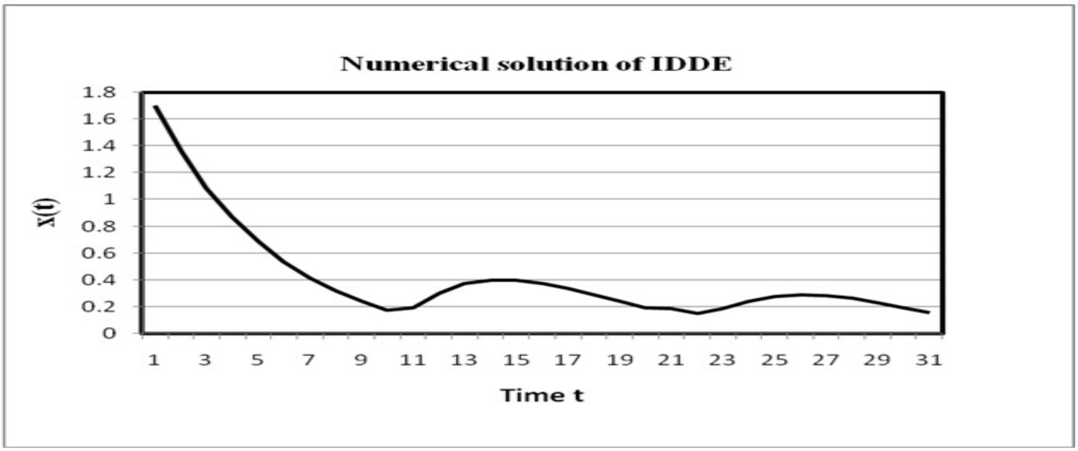

Figure 2. Diagramatic representation of numerical solution by Euler method of IDDE

Remark 2. From Figure 2, we can see that numerical simulation of the solution of same problem goes in same direction to that of analytic solution. This solution is also exponentially stable. It can also be noted that the error between both solutions near impulsive jumps is very small and can be improved by using better techniques for numerical solutions.

4. Conclusion

In this paper, we proposed a new approach for analytical solution of impulsive delay differential equations with impulse effect at fixed moments. We intercepted the algorithm following the theory of impulsive differential equations. Solving of Impulsive differential equations has not been explored by many researchers due to discontinuity at impulse moments and dependence of solution on its previous values. Therefore, this technique can be very useful in many studies to enhance and verify the existing results. From the diagrammatic comparison with the corresponding numerical solution of same equation it can be easily observed that both the solutions are moving in same direction and are exponentially stable. Further this method can also be adopted for comparative study with other numerical techniques such as Taylor series method, RK method of any order or with multistep methods. Moreover, this technique can be very useful in many studies to enhance and verify the existing results. This study can also be used to study the qualitative behavior of impulsive differential systems.

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